derived by using various other approaches; however, the objective of this section is to show the simplicity of applying the new approach to simple rotation about an arbitrary axis.

The remainder of this Note presents an application of this approach to the derivation of the transformation matrices of the space-axis rotation.

Space-Axis Rotation

The space-axis and body-axis rotations are defined here as a successive rotation about the space-fixed axes and body-fixed axes, respectively. A very interesting relationship between these different schemes of successive rotations has been discussed in Ref. 1. For example, if B is subjected successively to $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ rotations of amounts θ_1, θ_2, and θ_3, respectively, then one can achieve the same orientation by successive $\hat{b}_3$, $\hat{b}_2$, and $\hat{b}_1$ rotations of amounts θ_3, θ_2, and θ_1, respectively. This Note shows that using the approach presented in Eq. (1) will naturally results in such an intimate relationship between the space-axis and body-axis rotations, without requiring explicit determinations of those intermediate matrices given in Ref.1.

Consider a space-axis rotation in the sequence $\theta_1\hat{a}_1$, $\theta_2\hat{a}_2$, and $\theta_3\hat{a}_3$ ($\theta_1\hat{a}_1$ means an $\hat{a}_1$ rotation of amount θ_1). The total transformation matrix can be defined as

$$ {}^{A}C^{B} \triangleq {}^{A}C^{\bar{B}}\ {}^{\bar{B}}C^{\bar{\bar{B}}}\ {}^{\bar{\bar{B}}}C^{B} \tag{6}$$

To deal with the $\hat{a}_1$ rotation, let

$$ {}^{A}C^{\bar{B}} \triangleq C_1(\theta_1) \text{ where } C_1(\theta_1) \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{7}$$

Next, to construct a matrix ${}^{\bar{B}}C^{\bar{\bar{B}}}$ that characterizes the $\hat{a}_2$ rotation with the amount θ_2, let us use the approach discussed in the previous section:

$$ {}^{\bar{B}}C^{\bar{\bar{B}}} = {}^{\bar{B}}C^{A}\ \ C_2(\theta_2)\ \ {}^{A}C^{\bar{B}}$$

where

$$ C_2(\theta_2) \triangleq \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{8}$$

Combining Eqs. (7) and (8), we get

$$ {}^{A}C^{\bar{\bar{B}}} = {}^{A}C^{\bar{B}}[{}^{\bar{B}}C^{A}\ \ C_2(\theta_2)\ \ {}^{A}C^{\bar{B}}] = C_2(\theta_2)C_1(\theta_1) \tag{9}$$

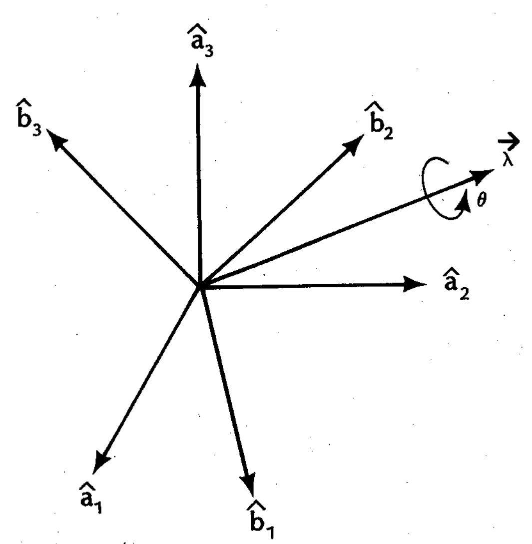

Fig. 1 Geometry of coordinate transformation and Euler's principal rotation.

Similarly, for the $\hat{a}_3$ rotation,we have

$$ {}^{\bar{\bar{B}}}C^{B} = {}^{\bar{\bar{B}}}C^{A}\ C_3(\theta_3)\ \ {}^{A}C^{\bar{\bar{B}}}$$

where

$$ C_3(\theta_3) \triangleq \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Finally, the total transformation matrix becomes

$$ {}^{A}C^{B} = {}^{A}C^{\bar{\bar{B}}}[{}^{\bar{\bar{B}}}C^{A}\ C_3(\theta_3)\ {}^{A}C^{\bar{\bar{B}}}] = C_3(\theta_3)C_2(\theta_2)C_1(\theta_1) \tag{11}$$

It can then be noticed that the total transformation matrix represented as Eq. (11) is, indeed, the transformation matrix for the successive $\hat{b}_3$, $\hat{b}_2$, and $\hat{b}_1$ rotations of θ_3, θ_2, and θ_1, respectively. Thus, the total transformation matrix due to the space-axis rotation (successive $\theta_1\hat{a}_1$, $\theta_2\hat{a}_2$, and $\theta_3\hat{a}_3$ rotations) is identical to the transformation matrix due to the body-axis rotation in the sequence of $\theta_3\hat{b}_3$, $\theta_2\hat{b}_2$, and $\theta_1\hat{b}_1$.

Although the total transformation matrix for the space-axis rotation has a simple form as Eq. (11), each intermediate transformation matrix is rather complicated as can be seen from Eqs. (8) and (10). However, the approach used here does not require an explicit determination of those intermediate matrices [e.g., see Eqs. (49) and (52) on p. 36 of Ref. (1)] to find the total transformation matrix. Indeed, an intimate relationship between two different rotation schemes has been obtained rather directly.

Conclusion

A new approach to the derivation of the coordinate transformation matrix for the space-axis rotation has been presented. The use of this approach has naturally resulted in an interesting relationship between the space-axis and body-axis rotations. This approach has also shown its simplicity for parameterizing the direction cosine matrix of the general rotation about an arbitrary axis. However, the practical significance of using the space-axis rotation instead of the commonly used body-axis rotation needs further study.

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Closed-Form Solution for a Class of Guidance Laws

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Introduction

THIS Note presents a closed-form solution of the equations of motion of an ideal missile pursuing a nonmaneuvering target according to a class of guidance laws cur-

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rently used and studied in homing missile design. The capture area, missile acceleration, and homing time duration are derived in a general form. As examples, the closed-form solutions of several guidance laws are obtained to demonstrate the effectiveness of the method presented here.

The closed-form solution of true proportional navigation (TPN) was first derived by Guelman[1] and generalized by Yang et al.[2] The work of Merrill led to a general concept of proportional navigation that can be described as follows. The general idea of the proportional navigation law for a homing missile is to turn the heading of the missile toward some desired direction as rapidly as possible by commanding missile accelerations that are proportional to the angular rate of such direction. Such a direction may be simply the line of sight,[4] the missile collision course,[3] or any other appropriate direction. In general, these heading directions may be expressed by

$$L = f(r,\theta)\boldsymbol{e}_r + g(r,\theta)\boldsymbol{e}_\theta \tag{1}$$

where r is the relative distance, θ the aspect angle, and $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ the unit vectors along and perpendicular to the line of sight (LOS), respectively. For reasons to become clear later, we assume F and G are functions of θ only; thus the missile acceleration command becomes

$$\begin{aligned} \boldsymbol{a}_m &= \dot{\boldsymbol{L}} \\ &= K[f'(\theta) - g(\theta)]\dot{\theta}\boldsymbol{e}_r + K[f(\theta) + g'(\theta)]\dot{\theta}\boldsymbol{e}_\theta \end{aligned} \tag{2}$$

where K is the proportionality constant and overdots denote the differentiation with respect to time t and primes the differentiation with respect to θ. In the next section, we will calculate the missile trajectories by solving the equations of motion with the missile acceleration given by Eq. (2) and express capture area, homing time duration, and missile acceleration commanded in terms of $f(\theta)$ and $g(\theta)$.

Closed-Form Solution

Let V denote the target velocity relative to the interceptor. In the polar coordinate system (r,θ) with the origin fixed at the interceptor, V is expressed as

$$\boldsymbol{V} = V_r\boldsymbol{e}_r + V_\theta\boldsymbol{e}_\theta = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta \tag{3}$$

From elementary kinematics, we have

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{a}_t - \boldsymbol{a}_m \tag{4}$$

where $\boldsymbol{a}_t$ is the target acceleration. A general analytical solution of Eq. (4) seems impossible; however, if $\boldsymbol{a}_t - \boldsymbol{a}_m$ can be approximated by the functional expression,

$$\boldsymbol{a}_t - \boldsymbol{a}_m = F(\theta)\dot{\theta}\boldsymbol{e}_r + G(\theta)\dot{\theta}\boldsymbol{e}_\theta \tag{5}$$

the closed-form solution will exist. This relation is always satisfied for the case where the missile acceleration is given by Eq. (2) and the target is nonmaneuvering. This allows us to evaluate analytically the performance of the various guidance laws characterized by Eqs. (1) and (2). Substituting $\boldsymbol{a}_t - \boldsymbol{a}_m$ and V from Eqs. (5) and (3), respectively, into Eq. (4), we have

$$\dot{V}_r - V_\theta\dot{\theta} = F(\theta)\dot{\theta} \tag{6a}$$

$$\dot{V}_\theta + V_r\dot{\theta} = G(\theta)\dot{\theta} \tag{6b}$$

To obtain the solution of this set of nonlinear equations, we change the independent variable from t to aspect angle θ and introduce new functions $g(\theta)$ and $f(\theta)$, which are related to $F(\theta)$ and $g(\theta)$ as

$$F(\theta) = -K[f'(\theta) - g(\theta)]$$

$$G(\theta) = -K[f(\theta) + g'(\theta)]$$

The constant K is introduced to make $f(\theta)$ and $g(\theta)$ dimensionless. By these substitutions we have

$$V_r' - V_\theta = -K[f'(\theta) - g(\theta)] \tag{7a}$$

$$V_\theta' + V_r = -K[f(\theta) + g'(\theta)] \tag{7b}$$

Note that Eqs. (7) can also be regarded as the equations of motion of an ideal missile pursuing a nonmanuevering target with the missile acceleration described by Eq. (2). The solution of Eqs. (7) with the initial conditions given by $V_r(\theta_0) = V_{r_0}$ and $V_\theta(\theta_0) = V_{\theta_0}$ can be obtained immediately as

$$-V_r = R\cos(\theta - \theta_0 + \alpha) + Kf(\theta) \tag{8a}$$

$$V_\theta = R\sin(\theta - \theta_0 + \alpha) - Kg(\theta) \tag{8b}$$

where

$$R = \sqrt{[V_{r_0} + Kf(\theta_0)]^2 + [V_{\theta_0} + Kg(\theta_0)]^2}$$

$$\tan(\alpha) = -\frac{Kg(\theta_0) + V_{\theta_0}}{Kf(\theta_0) + V_{r_0}}$$

Without loss of generality, we choose $\theta_0 = \alpha$ in the following discussion. The hodograph is determined by eliminating θ between Eqs. (8a) and (8b).

For the missile to intercept the target with a finite acceleration and with a finite time of duration, we must have $\dot{\theta} < \infty$ and $r = 0$, $\dot{r} < 0$ at the end of pursuit, i.e., $V_\theta(\theta_f) = 0$ and $V_r(\theta_f) < 0$. By means of Eqs. (8), these conditions are reduced to the form,

$$R\sin(\theta_f) - Kg(\theta_f) = 0 \tag{9a}$$

$$R\cos(\theta_f) + Kf(\theta_f) < 0 \tag{9b}$$

where θ_f is the aspect angle at the end of pursuit. By substituting for θ_f determined from Eq. (9a) in Eq. (9b), we can determine the range of initial conditions for V_{r_0} and V_{θ_0}, within which the interception can be completed. The inequality thus obtained from Eq. (9b) defines the so-called capture area. For a powerful guidance law, the capture area must be large, which poses restrictions on the choice of $f(\theta)$ and $g(\theta)$.

Define the angular momentum per unit mass for the relative coordinate system to be

$$h = rV_\theta \tag{10}$$

Then Eq. (7b) can be rewritten as

$$r = \frac{-Kh[f(\theta) + g'(\theta)]}{\mathrm{d}h/\mathrm{d}t}$$

Differentiating both sides of the above equation with respect to t and using Eqs. (9), we have the following important relation between h and θ:

$$\frac{\mathrm{d}h}{h} = -k\frac{f(\theta) + g'(\theta)}{R\sin(\theta) - Kg(\theta)}\mathrm{d}\theta \tag{11}$$

The integration of Eq. (11) gives

$$h = h_0\exp[-\Phi(\theta)] \tag{12}$$

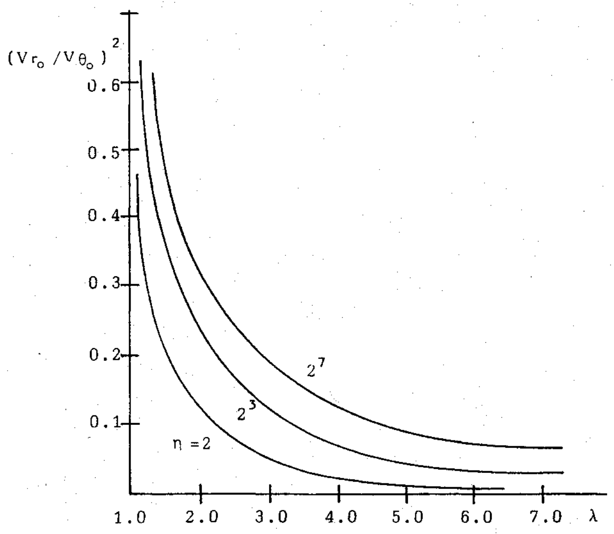

Fig. 1 Capture area for PRG.

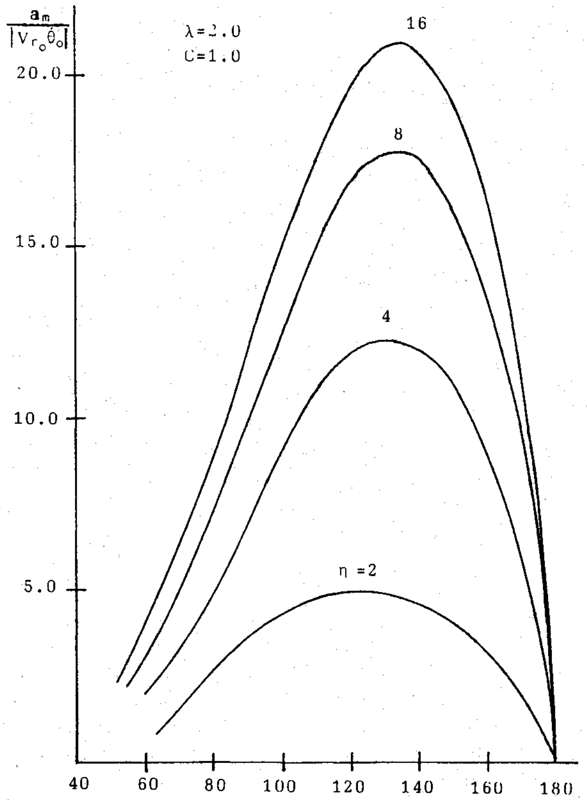

Fig. 2 Missile acceleration vs aspect angle for various values of η.

where $\Phi(\theta)$ is defined by

$$\Phi(\theta)=\int_{\theta_0}^{\theta} k\frac{f(\phi)+g'(\phi)}{R\sin(\phi)-kg(\phi)}\mathrm{d}\phi$$

Note that if $f(\theta)$ and $g(\theta)$ are trigonometric functions of θ, then $\Phi(\theta)$ can always be expressed by elementary or elliptical functions. Once h has been obtained, one can determine the missile acceleration. From Eq. (10), we get the LOS angular rate

$$\dot{\theta}=\frac{V_\theta^2}{h}=\frac{1}{h_0}[R\sin(\theta)-kg(\theta)]^2\exp[\Phi(\theta)] \quad (13)$$

Substituting this into Eq. (2), we have an expression for missile acceleration as a function of θ. From Eq. (13), it can be seen that θ is monotonically increasing or decreasing (depending on the sign of h_0) function of t. This also justifies the use of θ as an independent variable.

The relative distance r is found to be

$$r=h/V_\theta=h_0[R\sin(\theta)-kg(\theta)]^{-1}\exp[-\Phi(\theta)] \quad (14)$$

and the duration of the pursuit can be obtained from the integration of Eq. (14). This leads to

$$t_f=h_0\int_{\theta_0}^{\theta_f}[R\sin(\phi)-kg(\phi)]^{-2}\exp[-\Phi(\phi)]\mathrm{d}\phi \quad (15)$$

where θ_f is determined from Eq. (9a).

These explicit expressions for capture area, missile acceleration, and homing time duration are quite useful in the selection of guidance laws. Regardless of how $f(\theta)$ and $g(\theta)$ are chosen, these relations allow us to evaluate the performance of the missile immediately.

Examples

Three special forms of $g(\theta)$ and $f(\theta)$ have appeared in the literature, namely,

1) True proportional navigation law[4]

$$f(\theta)=1, \qquad g(\theta)=0 \quad (16)$$

2) Generalized proportional navigation law[1]

$$f(\theta)=\cos(\psi), \qquad g(\theta)=\sin(\psi) \quad (17)$$

where ψ is a constant angle between the missile acceleration and the direction normal to the LOS. When $\psi=0$, this reduces to case[1].

3) Prediction guidance law (PRG).[5] The main idea of PRG is to predict a straight-line collision course and to turn the heading of the missile toward a collision course as rapidly as possible. In the case having a nonmaneuvering target and a constant ratio of missile speed to target speed, this missile collision course can be predicted a priori. It can be shown that

$$f(\theta)=[\sqrt{\eta^2-\sin^2(\theta)}-\cos(\theta)]\eta, \quad g(\theta)=0 \quad (18)$$

where the direction of target velocity is used as the reference line and η is the missile-to-target speed ratio. If we let η approach infinity, PRG again reduces to TPN. The closed-form solution of case 1 was first derived by Guelman[1] and generalized in Ref. 2, where case 2 was studied and the results showed that, under certain conditions, the GPN has a larger capture area and a shorter interception time than TPN. For the case 3, a numerical simulation was given in Ref. 5 and the advantages of PRG were indicated. Based on the present closed-form study, further remarkable properties of PRG are revealed. The corresponding function $\Phi(\theta)$ for PRG can be found from Eq. (11) as

$$\Phi(\theta)=\frac{K}{\eta R}\ln|f(\theta)\sin(\theta)|+\frac{K}{2R}\ln\left|\frac{1-\eta^2/(\eta+1)^2f^2(\theta)}{1-\eta^2/(\eta-1)^2f^2(\theta)}\right| \quad (19)$$

where $f(\theta)$ is defined in Eq. (18).

By substituting these functions in Eqs. (13-15), the missile acceleration, relative distance, and homing time duration can be found immediately. In the following, we will give some results for PRG; refer to Refs. 1 and 2 for detailed discussions for the closed-form solutions of TPN and GPN.

The capture area of PRG is given by Eq. (10) as

$$[\lambda f(\theta_0)-1]^2+(1/C)<\lambda^2(1+1/\eta)^2 \tag{20}$$

where $\lambda=-K/V_{r_0}$ and $C=|V_{r_0}/V_{\theta_0}|$.

This is depicted in Fig. 1. The capture area of the missile is restricted to the right side of each curve representing different values of η. It can be seen that the larger η is, the smaller the capture area will be and that TPN ($\eta \rightarrow \infty$) has the smallest capture area. Notice that in proportional navigation η is indeed not a constant value and that here we may treat η as the average missile-to-target speed ratio or merely as a design parameter. For commanded missile acceleration, we prefer the PRG. It can be seen from Fig. 2 that missile acceleration becomes larger as η increases. Finally, the homing time duration obtained from Eq. (15) shows that it approaches a constant as η increases without bound. We then conclude that, for a proper choice of η (neither too small nor too large), PRG is superior to TPN, at least with respect to the factors considered above.

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